#16: Fracture: Global Analysis

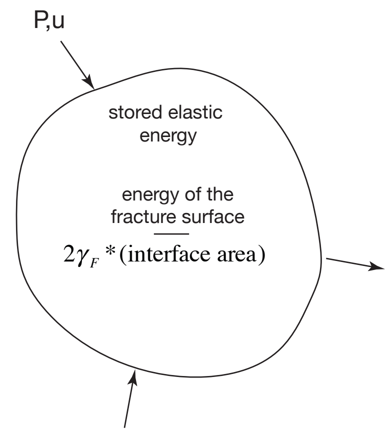
Contributions to Energy in the System

Global energy analysis of fracture usually includes three elements:

(i) The work done by external forces and displacements (it is equal to the product of the forces and the displacements).

(ii) The change in the stored elastic energy in body (system)

(iii) The work of fracture which is given by the product of the work done per unit area to create the fracture ($2\gamma_F$) times the total surface area of the fracture surface.



Example: A sample held in tension under a uniaxial load

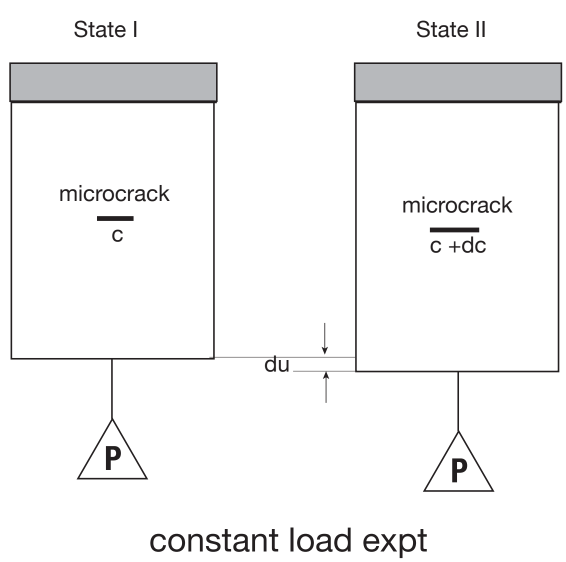
The question: the sample has a small microcrack, and we wish to know the critical stress under which the crack will propagate.

The Method-1: incremental crack growth

We consider two states, where in one case the crack has a size of c and in the other a size of $c+dc$.

•Since the crack surface area has increased work has been done to extend the fracture

•Now we must consider whether or not the work of fracture can be supplied by external work (force time displacement) and/or the change in the stored elastic energy in the system.



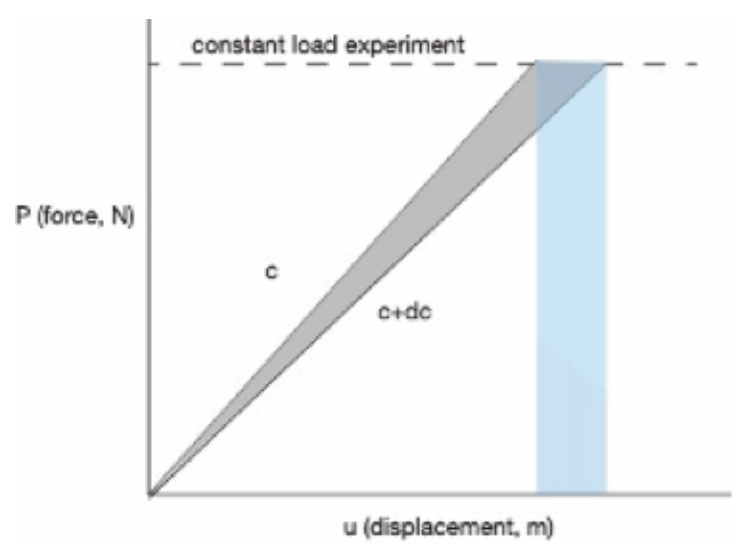


Figure 1. Assume the specimen to have length, L, perpendicular to the plane of the paper. Now consider the difference in the three components of energy between State II and State I.

These are three contributions are shown below. Remember we are describing the work in traversing from State I to State II. If the energy of State II is higher then the change is positive, if it is lower then it is negative.

(i) Increase in the work of fracture

$+2\gamma_F L * dc$

(ii) Work done by the applied load on the system

$-P * du$

(iii) Change in the stored elastic energy

•The stored elastic energy *increases* in going from State I to State II

•The stored elastic cnergy = $\frac{Pu}{2}$, i. e. the area in the triangles shown above.

•The area for the triangle in State II is larger than the triangle in State I

•From geometry we can assert that this difference is equal to + $\frac{Pdu}{2}$, i.e. it is equal to one half of the rectangle that represents the change in the potential energy of the dead weight.

Combining (ii) and (iii)

The sum of the change in the potential energy and the stored elastic energy = $-P * du + \frac{P * du}{2}$; that is, one is exactly one half of the other. The total change in the mechanical energy when the crack advances by du is (i) one half of the decrease in the potential energy, or it can also be said that (ii) it equal to the change in the stored elastic energy.

Putting it together

Criterion of crack propagation is that the work need to propagate the crack (from State I to State II) must be less than or equal to the mechanical energy available.

$2\gamma_F L * dc - \frac{P * du}{2} \leq 0$. We simply say that the critical condition is $2\gamma_F L * dc = \frac{P * du}{2}$

$$2\gamma_F = \frac{P}{2L}\frac{du}{dc} \qquad (1)$$

$S = \frac{u}{P}$ where S is the compliance, it is the inverse of the slope of the lines drawn in the figure above.

$$\frac{dS}{dc} = \frac{du}{dc}\frac{1}{P} \qquad (2)$$

If combine (1) and (ii)

$$2\gamma_F = \frac{P^2}{2L}\frac{dS}{dc} \qquad (3)$$

$2\gamma_F$ is the work of fracture and it is a material parameter. Here P is the critical load for crack propagation. L is sample geometry. $\frac{dS}{dc}$ is the change in compliance with crack length. It can be experimentally determined by making specimens with different crack lengths and measuring their compliance and then making of plot of compliance against the crack length.

Method-II: Gibbs Free Energy Approach

Gibbs free energy is a "thermodynamic" quantity that is used to describe reaction equilibria, that is, whether or not a certain event, usually electro-chemo-mechanical in nature, is possible. For example it can describe he amount of salt that will saturate water at a given temperature.

Gibbs free energy is the total work done when a system changes from one state to another state, including the work done by the surroundings on the system and the work done within the system.

If this work (in changing from State I to State II) is positive then State I will remain stable. If this work is negative then State II will prevail.

It is important to keep in mind the sign convention for the change in an element of energy between State II and State I. (a) If the surroundings do work on the system in traversing from State I to State II, for example if the potential energy of State II is lower than in Stage I then the change in Free Energy is negative. (b) If the internal energy of the system in State II is higher than in State I then the change in Gibbs Free energy is positive. And so on.

With the above in mind let us consider the problem in Figure 1 in a different way

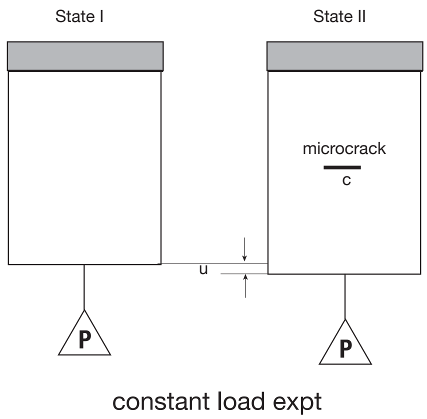


Figure 2: Note the difference between Fig. 1 and Fig. 2. In the first we are considering incremental growth in the crack, while in the second we are assuming the crack to be absent in State I and present in Stage II.

In this second method we consider how the difference in the Gibb free energy is affected by the size of the crack. Now consider how ΔG changes with crack size, and then consider this rate of change to find the point of instability. Please also note that the crack is now in the shape of a penny.

$$\Delta G = -Pu + \frac{\pi}{4}c^2(2\gamma_F) + \frac{Pu}{2}$$
$$\frac{d\Delta G}{dc} = -\frac{P}{2}\frac{du}{dc} + \pi c\gamma_F \qquad (4)$$

Borrowing from Eq. (2) to substitute for $\frac{du}{dc} = P\frac{dS}{dc}$ into Eq. (4) we get

$$\frac{d\Delta G}{dc} = -\frac{P^2}{2}\frac{dS}{dc} + \pi c\gamma_F \qquad (5)$$

In Eq. (4) at the top: •the first term is the work done on the system by the surroundings represented by the drop in the potential energy of the dead weight, •the second term is the energy of the fracture surface of the penny shaped crack, and the •third term is the increase in the stored energy in State II relative to State I.

When plotting ΔG against c, Eq. (4) will have a maximum where the slope is zero, beyond which the slope becomes negative that is the free energy declines as the crack grows: this is the point of instability. Therefore the instability to fracture is given by equating (5) to zero,

$$2\gamma_F = \frac{P^2}{\pi c}\frac{dS}{dc} \qquad (6)$$

Let us check the units. P has units of N, S has units of Nm^{-1}, and c of course has units of m. So the right hand side has units of $N^2(mN^{-1})m^{-2} = (Nm)/m^2$, i. e. energy per unit area.

Note how the result in Eq. (6) differs from Eq. (2) because we are consider a penny shaped crack , of diameter c, instead of a ribbon shaped crack of length L. Therefore $2L$ becomes replaced by πc.